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Higher-dimensional Generalisations of the Euler Top Equations

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ABSTRACT

Generalisations of the familiar Euler top equations in three dimensions are proposed which admit a sufficiently large number of conservation laws to permit integrability by quadratures. The usual top is a classical analogue of the Nahm equations. One of the examples discussed here is a seven-dimensional Euler top, which arises as a classical counterpart to the eight-dimensional self-dual equations which are currently believed to play a role in new developments in string theory.

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1 Introduction

The object of this note is to examine the question of the existence of Euler equations in higher dimension which are integrable generalisations of the familiar Euler equation for a top, with quadratic nonlinearities. The motivation for this last requirement comes from the fact that the resulting equations may be viewed as classical versions of the Nahm equations [1]. The higher-dimensional equivalents of the Nahm equations, i.e. self-dual Yang-Mills equations with gauge fields dependent only upon time, which have appeared recently in the context of M-Theory [2][3][4][5][6][7], might well be expected to give rise to a similar classical reduction. We shall find that there is indeed an exact parallel, satisfying a similar Bogomol'nyi bound which is integrable by quadratures, just as in the three-dimensional case, but this time it is integrable only in principle. There are sufficient conservation laws to permit full integrability, but their algebraic solution is prohibitive in practice. As a by-product a remarkable identity among seven quartic polynomials, which reduces the number of conserved quantities to permit a non constant solution, is discovered.

1.1 Three-dimensional example

The standard Euler top in three dimensions in its simplest form may be written as

$$\dot{\omega}_1 = \omega_2\,\omega_3 \ , \tag{1}$$

together with cyclic replacements. This set of equations results when a solution of the three-dimensional Nahm equations,

$$\frac{dA_i}{dt} = \frac{1}{2}\epsilon_{ijk}[A_j,\ A_k] \ , \tag{2}$$

is sought in the form, in terms of the Pauli matrices σ_i,

$$A_i = \omega_i\,\sigma_i \ , \qquad (i \text{ not summed.}) \tag{3}$$

From (1), we have two independent conserved quantities,

$$\omega_2^2 - \omega_1^2 = c_2 \ , \quad \omega_3^2 - \omega_1^2 = c_3 \ . \tag{4}$$

As well known, the general solution of (1) is given by elliptic functions. Solving (4) for ω_2 and ω_3 and substituting them into the equation for ω_1 in (1), we have

$$\dot{\omega}_1 = \sqrt{(\omega_1^2 + c_2)(\omega_1^2 + c_3)} \ . \tag{5}$$

Parametrising the constants c_2 and c_3 as $c_2 = \alpha^2 k'^2 = \alpha^2(1-k^2)$ and $c_3 = \alpha^2$, we have the solution of the equation,

$$(\omega_1(t), \omega_2(t), \omega_3(t)) = (\alpha k'\, tn(\alpha t + \beta), \alpha k'\, nc(\alpha t + \beta), \alpha\, dc(\alpha t + \beta)) \ , \tag{6}$$

where $tn(x)$, $nc(x)$ and $dc(x)$ are elliptic functions and β is a constant. Note that (1) is invariant under the change of sign of any two of ω_i and hence, in addition to the above solution, of $(+,+,+)$ type say, we have other $(+,-,-)$ and $(-,\pm,\mp)$ type solutions.

2 Seven-dimensional top

A similar procedure applied to the equivalent of the Nahm equations for self-dual fields in eight dimensions [2][5][6], which is defined using the octonionic structure constants C_{ijk} $(i,j,k = 1,\cdots,7)$ in (2) instead of the quaternionic ones ϵ_{ijk}. For an explicit realisation of the totally anti-symmetric C_{ijk},

$$C_{127} = C_{631} = C_{541} = C_{532} = C_{246} = C_{734} = C_{567} = 1 \ , \qquad \text{(others zero)} \tag{7}$$

we have the set of seven equations;

$$\begin{aligned}
&\frac{d}{dt}A_1 - [A_2,\ A_7] - [A_6,\ A_3] - [A_5,\ A_4] = 0 \ , \\
&\frac{d}{dt}A_2 - [A_7,\ A_1] - [A_5,\ A_3] - [A_4,\ A_6] = 0 \ , \\
&\frac{d}{dt}A_3 - [A_1,\ A_6] - [A_2,\ A_5] - [A_4,\ A_7] = 0 \ , \\
&\frac{d}{dt}A_4 - [A_1,\ A_5] - [A_6,\ A_2] - [A_7,\ A_3] = 0 \ , \\
&\frac{d}{dt}A_5 - [A_4,\ A_1] - [A_3,\ A_2] - [A_6,\ A_7] = 0 \ , \\
&\frac{d}{dt}A_6 - [A_3,\ A_1] - [A_2,\ A_4] - [A_7,\ A_5] = 0 \ , \\
&\frac{d}{dt}A_7 - [A_1,\ A_2] - [A_3,\ A_4] - [A_5,\ A_6] = 0 \ .
\end{aligned} \tag{8}$$

Note that the gauge field in the t direction has been set to zero to obtain these equations. This generalisation of the Euler top is rather different from that proposed by Manakov [8]. A search for a solution in the form

$$A_i = \omega_i \, e_i \ , \qquad (i \text{ not summed,}) \tag{9}$$

where e_i are a basis for octonions, yields the following equations;

$$\begin{aligned}
&\frac{d}{dt}\omega_1 - \omega_2\omega_7 - \omega_6\omega_3 - \omega_5\omega_4 = 0 \ , \\
&\frac{d}{dt}\omega_2 - \omega_7\omega_1 - \omega_5\omega_3 - \omega_4\omega_6 = 0 \ , \\
&\frac{d}{dt}\omega_3 - \omega_1\omega_6 - \omega_2\omega_5 - \omega_4\omega_7 = 0 \ , \\
&\frac{d}{dt}\omega_4 - \omega_1\omega_5 - \omega_6\omega_2 - \omega_7\omega_3 = 0 \ , \\
&\frac{d}{dt}\omega_5 - \omega_4\omega_1 - \omega_3\omega_2 - \omega_6\omega_7 = 0 \ , \\
&\frac{d}{dt}\omega_6 - \omega_3\omega_1 - \omega_2\omega_4 - \omega_7\omega_5 = 0 \ , \\
&\frac{d}{dt}\omega_7 - \omega_1\omega_2 - \omega_3\omega_4 - \omega_5\omega_6 = 0 \ .
\end{aligned} \tag{10}$$

The structure of these equations can be read off from Fig.1, the seven-point plane. This construction arises from the projective geometry of a plane over a finite field of characteristic two; 3 points lie on every line and 3 lines pass through each point. The contributions to $\dot{\omega}_i$ come from the products of the pairs of ω's associated with the other points on each of the three lines through i.

3 Solutions

This set of equations admits the trivial symmetric solution

$$\omega_i = \frac{-1}{3t} \ , \tag{11}$$

and also several broken symmetry solutions, e.g.

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \frac{-1}{t} \ , \quad \omega_5 = \omega_6 = \frac{1}{t} \ , \quad \omega_7 = \frac{-3}{t} \ . \tag{12}$$

The second order equations implied by (10) come from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\sum_{i=1}^{7}(\dot{\omega}_i)^2 + \sum_{i=1}^{7}\sum_{j<i}\omega_i^2\omega_j^2\right) + 3\sum_{\text{perms}}\omega_i\omega_j\omega_k\omega_l \ , \tag{13}$$

where the last sum in the above expression is taken over all distinct values of $\omega_i\omega_j\omega_k\omega_l$ corresponding to indices $i,\ j,\ k,\ l$ no three of which lie on a line. The Lagrangian is given up to a divergence by the sum of the squares of the equations (10). A similar Bogomol'nyi property is known for the commutator case [5][9].

3.1 Conservation Laws

The equations (10) can be rewritten as

$$\begin{aligned}
\frac{d}{dt}(\omega_1 \pm \omega_2) &= (\omega_6 \pm \omega_5)(\omega_3 \pm \omega_4) \pm (\omega_1 \pm \omega_2)\omega_7 \ , \\
\frac{d}{dt}(\omega_6 \pm \omega_5) &= (\omega_3 \pm \omega_4)(\omega_1 \pm \omega_2) \pm (\omega_6 \pm \omega_5)\omega_7 \ , \\
\frac{d}{dt}(\omega_3 \pm \omega_4) &= (\omega_1 \pm \omega_2)(\omega_6 \pm \omega_5) \pm (\omega_3 \pm \omega_4)\omega_7 \ ,
\end{aligned} \tag{14}$$

where either the plus or minus sign is taken consistently throughout each equation. The two sets of equations with plus and minus signs are linked by the seventh equation

$$\frac{d}{dt}\omega_7 = \frac{1}{4}\left((\omega_1+\omega_2)^2 + (\omega_6+\omega_5)^2 + (\omega_3+\omega_4)^2 - (\omega_1-\omega_2)^2 - (\omega_6-\omega_5)^2 - (\omega_3-\omega_4)^2\right) . \tag{15}$$

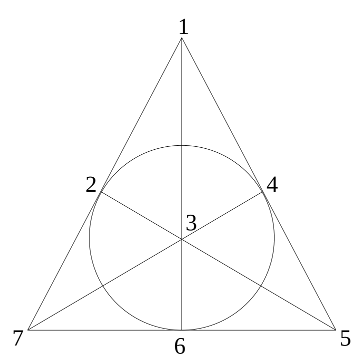

Figure 1: 7 point plane.

Integrals of motion, obtained from these equations are of two types;

$$\frac{(\omega_i \pm \omega_j)^2 - (\omega_k \pm \omega_l)^2}{(\omega_i \pm \omega_j)^2 - (\omega_m \pm \omega_n)^2} = \alpha_{hikm\pm} \ , \tag{16}$$

and

$$((\omega_k + \omega_l)^2 - (\omega_m + \omega_n)^2)((\omega_k - \omega_l)^2 - (\omega_m - \omega_n)^2) = \beta_{hij} \ , \tag{17}$$

where $h,\ i,\ j,\ k,\ l,\ m,\ n$ is a permutation of the indices $1\ldots 7$ such that $(i,\ j)$, $(k,\ l)$ and $(m,\ n)$ lie on the respective three lines through h in Fig.1 and $\alpha_{hikm\pm}$, β_{hij} are constants of integration. There are several identities among these relationships, reducing their number. For example,

$$\alpha_{hikm+}\alpha_{hikm-}\alpha_{hmik+}\alpha_{hmik-}\alpha_{hkmi+}\alpha_{hkmi-} = 1 \ , \tag{18}$$

and similarly

$$\alpha_{hikm\pm} = \alpha^{-1}_{himk\pm} \ , \qquad \alpha_{hikm\pm} + \alpha_{hmki\pm} = 1 \ . \tag{19}$$

There are still apparently too many. We need to reduce the number of independent conserved quantities to six, to enable a proper dynamical evolution to take place. This can be achieved as follows, by concentrating upon the second set, which can be re-expressed as

$$(\omega_k + \omega_l + \omega_m + \omega_n)(\omega_k + \omega_l - \omega_m - \omega_n)(\omega_k - \omega_l + \omega_m - \omega_n)(\omega_k - \omega_l - \omega_m + \omega_n) = \beta_{hij} \ , \tag{20}$$

showing that only 7 of the above expressions are different. Remarkably, the Jacobian of the seven quantities in (20) with respect to the variables ω_j is found to vanish, showing that they are functionally dependent. It is also found, by computer calculations that the generic rank of this Jacobian is 6, showing that there are 6 functionally independent conserved quantities. The explicit functional relation between the 7 quartic polynomials, say $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7) = (\beta_{567}, \beta_{631}, \beta_{734}, \beta_{541}, \beta_{532}, \beta_{127}, \beta_{246})$, is given by

$$\sum_{(i,j,k)} \gamma_i\gamma_j\gamma_k \, [(\sum_{p=1}^{7} \gamma_p - 2(\gamma_i + \gamma_j + \gamma_k))^2 + \sum_{p=1}^{7} \gamma_p^2 - 2(\gamma_i^2 + \gamma_j^2 + \gamma_k^2)] = 0 \ , \tag{21}$$

where the sum is taken over all indices (i, j, k) corresponding to points lying on lines in Fig.1. Furthermore, there are no further independent quantities, since any of $\alpha_{hikm\pm}$ in the set (16) can be shown to be a solution of one of the following quadratic equations,

$$\gamma_i \, x^2 - (\gamma_i + \gamma_j - \gamma_k)x + \gamma_j = 0 \ , \tag{22}$$

where as before, the indices (i, j, k) lie on lines in Fig.1, showing that only 6 relations are functionally independent. Thus we can *in principle* solve for $\omega_j,\ j > 1$ in terms of ω_1, substitute in (10) and thus obtain an implicit integral solution of this set of equations.

4 Other generalisations

As was noted in the previous section, the equations (10) reduce to two linked sets of three-dimensional equations. If we set $\omega_7 = 0$ and remove the final equation, then the set;

$$\begin{aligned}
&\frac{d}{dt}\omega_1 - \omega_6\omega_3 - \omega_5\omega_4 = 0 \ , &\qquad &\frac{d}{dt}\omega_2 - \omega_5\omega_3 - \omega_4\omega_6 = 0 \ , \\
&\frac{d}{dt}\omega_3 - \omega_1\omega_6 - \omega_2\omega_5 = 0 \ , &\qquad &\frac{d}{dt}\omega_4 - \omega_1\omega_5 - \omega_6\omega_2 = 0 \ , \\
&\frac{d}{dt}\omega_5 - \omega_4\omega_1 - \omega_3\omega_2 = 0 \ , &\qquad &\frac{d}{dt}\omega_6 - \omega_3\omega_1 - \omega_2\omega_4 = 0 \ ,
\end{aligned} \tag{23}$$

decouples into two independent Euler top equations;

$$
\begin{aligned}
\frac{d}{dt}(\omega_1 \pm \omega_2) &= (\omega_6 \pm \omega_5)(\omega_3 \pm \omega_4) \ , \\
\frac{d}{dt}(\omega_6 \pm \omega_5) &= (\omega_3 \pm \omega_4)(\omega_1 \pm \omega_2) \ , \\
\frac{d}{dt}(\omega_3 \pm \omega_4) &= (\omega_1 \pm \omega_2)(\omega_6 \pm \omega_5) \ .
\end{aligned}
\tag{24}
$$

Thus the system is completely integrable in terms of elliptic integrals.

4.1 Back to seven dimensions

Another seven-dimensional set of equations can be constructed which provides an interesting example of a system which is partially integrable, but probably not fully integrable.

This example is an extension of a simple idea for a generalisation of the Euler equations in [10]. The idea in that paper is that the generalisation to n variables ω_i is to take the n equations,

$$\dot{\omega}_i = \prod_{j \neq i}^{n} \omega_j \ . \tag{25}$$

Then it is easy to verify that the quantities,

$$\omega_i^2 - \omega_j^2 = d_j - d_i \ , \tag{26}$$

where d_i are constants, are $n-1$ independent constants of the motion which can be solved to express $\omega_j, \ j > 1$ in terms of ω_1, giving a differential equation for ω_1 integrable by quadratures in terms of hyperelliptic functions, of the form,

$$\dot{\omega}_1 = \sqrt{\prod_{j=2}^{n} (\omega_1^2 - d_j)} \ , \tag{27}$$

where d_1 has been set to zero for aesthetic reasons. Now this result can be used to find an integrable subset of solutions of a related set of equations with quadratic nonlinearities, namely

$$\frac{d}{dt} y_{ij} = \sum_{k \neq i,j}^{n} y_{ik} y_{jk} \ . \qquad (i \neq j) \tag{28}$$

Through the ansatz

$$y_{ij} = \prod_{k \neq i,j}^{n} \omega_k \ , \qquad (i \neq j) \tag{29}$$

it may be readily verified that a solution to (28) is given in terms of the solution to (25), which has been already discussed in [10]. The $n = 4$ case in (28), with $y_{ij} = y_{ji}$, is exceptional. This is nothing but the set of equations (23) and is completely integrable. Hence general solutions of four-dimensional Euler top equations in (25) are given by elliptic functions.

The number of variables for which this stratagem works is $\frac{1}{2}n(n-1)$. What we want to do is to realise a similar connection for seven-dimensional equations. Consider the equations of motion;

$$\begin{aligned}
\frac{d}{dt}y_1 &= y_2y_4 + y_6^2 + y_3y_7 \ , \\
\frac{d}{dt}y_2 &= y_5y_1 + y_3^2 + y_6y_7 \ , \\
\frac{d}{dt}y_3 &= y_4y_6 + y_2^2 + y_5y_7 \ , \\
\frac{d}{dt}y_4 &= y_1y_3 + y_5^2 + y_2y_7 \ , \\
\frac{d}{dt}y_5 &= y_6y_2 + y_4^2 + y_1y_7 \ , \\
\frac{d}{dt}y_6 &= y_3y_5 + y_1^2 + y_4y_7 \ , \\
\frac{d}{dt}y_7 &= y_1y_2 + y_3y_4 + y_5y_6 \ .
\end{aligned} \tag{30}$$

Then the following slight extension of the three-dimensional Euler equations;

$$\dot{\omega}_1 = \omega_2^2\omega_3^2 \ , \tag{31}$$

together with cyclic replacements can be related to the equations (30) by defining the following seven quantities,

$$y_1 = \omega_1^2\omega_2 \ , \ y_3 = \omega_2^2\omega_3 \ , \ y_5 = \omega_1\omega_3^2 \ , \ y_2 = \omega_1\omega_2^2 \ , \ y_6 = \omega_1^2\omega_3 \ , \ y_4 = \omega_2\omega_3^2 \ , \ y_7 = \omega_1\omega_2\omega_3 \ . \tag{32}$$

A solution of (31) will give rise to a solution of the set of seven equations with quadratic nonlinearities (30). Such a solution may be constructed by quadratures since it is easy to see that

$$\omega_2^3 = \omega_1^3 - c_2 \ , \quad \omega_3^3 = \omega_1^3 - c_3 \ , \quad (c_2, \ c_3, \ \text{constant}) \tag{33}$$

and hence the equations can be solved in terms of ω_1 and thus, in principle the equations (31) are reduced to integrations. We have been able to find only one quartic conserved quantity for the set (30), namely

$$\begin{aligned}
&(-y_1y_4 + y_2y_5 + y_3y_6 - y_7^2)^2 - 4(y_3y_5 - y_4y_7)(y_2y_6 - y_1y_7) = \\
&(y_1y_4 + y_2y_5 - y_3y_6 - y_7^2)^2 - 4(y_2y_4 - y_3y_7)(y_1y_5 - y_6y_7) = \\
&(y_1y_4 - y_2y_5 + y_3y_6 - y_7^2)^2 - 4(y_4y_6 - y_5y_7)(y_1y_3 - y_2y_7) = d \ ,
\end{aligned} \tag{34}$$

and it appears likely that the full system is not integrable, when the constraints implied by (32), namely

$$y_1y_4 = y_2y_5 = y_7^2 \ , \qquad y_1y_3 = y_2y_7 \ , \qquad y_2y_6 = y_1y_7 \ , \tag{35}$$

are relaxed.

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